MATHEMATICS

EXPONENTIAL SUMS DEVIATING LEAST FROM ZERO

BY

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Summary

In this paper exponential sums of the type $Y_1e^{w_1x}+Y_2e^{w_2x}+\ldots+Y_ne^{w_nx}$ where $w_1, w_2, \ldots, w_n$ are pairwise distinct real numbers and $Y_1, Y_2, \ldots, Y_n$ are polynomials in a real variable x of degree $\mu_1, \mu_2, \ldots, \mu_n$ (non-negative integers) respectively are considered. The problem posed is minimizing these sums in Tchebycheff norm on an interval $[-a, a]$ while satisfying the so called condition C (defined in section 1); the existence of a unique solution is then proved using the notion of a Tchebycheff-system (section 2).

At last convergence results for $\mu_1+\mu_2+\ldots+\mu_n \to \infty$ are derived from results in the theory of the n-dimensional Padé-table for the functions $e^{w_1x}, e^{w_2x}, \ldots, e^{w_nx}$ (section 3).

INTRODUCTION

In this paper exponential sums are considered as they arise in multi-dimensional Padé approximation. In a classical paper "Sur la généralisation des fractions continues algebriques" H. PADÉ [7] posed the following problem:

Determine polynomials $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ in the indeterminate x, not all identically zero, with complex coefficients and degrees at most $\mu_1, \mu_2, \ldots, \mu_n$ (non-negative integers) respectively, such that

$$X_1S_1+X_2S_2+\ldots+X_nS_n=0(x^{\mu_1+\mu_2+\ldots+\mu_n+n-1})$$

where $S_1, S_2, \ldots, S_n$ are given power series in x all with non-zero constant term and where $0(x^{\mu_1+\mu_2+\ldots+\mu_n+n-1})$ denotes a power series in x in which the coefficients of $x^0, x^1, \ldots, x^{\mu_1+\mu_2+\ldots+\mu_n+n-2}$ all vanish.

Under certain additional conditions the so-called Padé-n-tuple $(X_1, X_2, \ldots, X_n)$ is uniquely determined. It is placed in a cell with coordinates $(\mu_1, \mu_2, \ldots, \mu_n)$. The configuration of all these cells forms the n-dimensional Padé-table for the power series $(S_1, S_2, \ldots, S_n)$.

These n-dimensional Padé-tables did not receive much attention hitherto if $n>2$ except for the treatment in the thesis of H. JAGER [5]. The case where $n=2$ is equivalent with the well-known Padé-table of rational fractions approximating a given power series in the Padé sense. For a treatment of this case see O. PERRON [10], H. S. WALL [15]. In a few cases the n-dimensional Padé-table is known explicitly and we focus our attention on the case where $S_k=e^{w_kx}$ $(k=1, 2, \ldots, n)$ with

$$w_1, w_2, \ldots, w_n \in \mathbb{C} \text{ and } w_k \neq w_l \text{ if } k \neq l \ (k, l=1, 2, \ldots, n).$$

In general an expression of the type $Y_1e^{w_1x}+Y_2e^{w_2x}+\ldots+Y_ne^{w_nx}$ where the $w_1, w_2, \ldots, w_n$ are pairwise distinct and $Y_1, Y_2, \ldots, Y_n$ are polynomials in x of degree $\mu_1, \mu_2, \ldots, \mu_n$ respectively is called an exponential sum of degree $\sum_{k=1}^n (\mu_k+1)-1$. These sums play an important rôle in non-linear-approximation; here they appear in linear approximation problems.

We see that in n-dimensional Padé-approximation to the set of functions $\{e^{w_kx}\}_{k=1}^n$ special exponential sums arise deviating least from zero in the Padé sense, i.e. an approximation in the space of formal power series with "valuation metric". In section 1 we give a brief introduction to multi-dimensional Padé-approximation while in section 2 the notion of a multi-dimensional Walsh-array is introduced. This is a table analogous to the n-dimensional Padé-table but now the polynomials $X_1, X_2, \ldots, X_n$ are determined, satisfying condition C of the next page, in such a way that $X_1S_1+X_2S_2+\ldots+X_nS_n$ is minimized in the Tchebycheff norm on a real interval; in our case the interval $[-a, a]$, $0<a<\min\{\varrho_1, \varrho_2, \ldots, \varrho_n\}$, where ϱ_i denotes the radius of convergence of S_i $(i=1, 2, \ldots, n)$. In the case of multi-dimensional approximation to the set $\{e^{w_kx}\}_{k=1}^n$ we prove convergence properties of the n-dimensional Padé-table and the corresponding n-dimensional Walsh-array. This is done in section 3.

1. Let F denote the space of all formal power series in an indeterminate x (i.e. all series of the form $\sum_{n=m}^{\infty} a_nx^n$, m an integer and $a_m\neq 0$), with complex coefficients. Addition, scalar multiplication, multiplication and division are defined as usual.

First we define an absolute value $|\ |_F$ on F by:

$$|A|_F=e^{-m} \text{ when } A=\sum_{n=m}^{\infty} a_nx^n \text{ with } a_m\neq 0.$$

The aforementioned "valuation metric" ϱ is then defined as:

$$\varrho(A, B)=|A-B|_F,\ A, B\in F.$$

Now let $S_k\in F$ with $|S_k|_F=1$ $(k=1, 2, \ldots, n)$ be n given elements of F. We ask for n polynomials (also elements of F) $X_1, X_2, \ldots, X_n$ of degree $\mu_1, \mu_2, \ldots, \mu_n$ at most, respectively, such that

$$\left|\sum_{k=1}^{n} X_kS_k\right|_F\leqslant e^{-\mu_1-\mu_2-\ldots-\mu_n-n+1}.$$

From H. Jager [5], theorem 1.1.1, page 8, the existence of an n-tuple of polynomials like that follows immediately, while after specialisation to $S_k=e^{w_kx}$ $(k=1, 2, \ldots, n$; elements of F satisfying $|S_k|_F=1$!) his theorems 1.2.1, 1.1.2 and 1.1.4 (page 13, 10 and 12 respectively) with $m=n$, $\varrho_k=\mu_k+1$ $(k=1, 2, \ldots, n)$ and $R=\mathbb{C}$ lead to:

Theorem 1.1. The problem of finding polynomials $X_1, X_2, \ldots, X_n$

in x of degree $\mu_1, \mu_2, \ldots, \mu_n$ at most respectively such that

$$\left|\sum_{k=1}^{n} X_k e^{w_k x}\right|_F \leqslant e^{-\mu_1-\mu_2-\ldots-\mu_n-n+1}$$

has a unique solution (apart from a multiplicative constant) which also satisfies:

i) $\left|\sum_{k=1}^{n} X_k e^{w_k x}\right|_F = e^{-\mu_1-\mu_2-\ldots-\mu_n-n+1}$.

ii) the degree of X_k is μ_k $(k=1, 2, \ldots, n)$.

Remark: A set of power series admitting a unique solution (apart from a multiplicative constant) for the aforementioned problem satisfying i) and ii) is called *perfect*, a quite natural generalization of the notion normal in the case of the ordinary Padé-table.

Imposing one further condition on the coefficients of the polynomials appearing in the theorem we find a unique solution and place it in an n-dimensional table as described in the introduction.

The condition will be referred to as:

CONDITION C: the coefficient of x^{μ_1} in the polynomial X_1 is $1/\mu_1!$

Remark: In order to obtain a really unique solution it is sufficient either to fix one of the coefficients of the Padé-polynomials or to fix the first non-zero coefficient of $\sum_{k=1}^{n} X_k(\mu_1, \mu_2, \ldots, \mu_n; x)e^{w_k x}$. Taking the last mentioned coefficient to be $\alpha/(\mu_1+\mu_2+\ldots+\mu_n+n-1)!$ ($\alpha \in \mathbb{C}$, "symmetric" condition), the coefficient of x^{μ_1} in X_1 will be $\alpha/\mu_1! \prod_{\nu=2}^{n} (w_1-w_\nu)^{\mu_\nu+1}$ *and conversely*.

Imposing a condition on the coefficient of the highest power of x appearing in X_1 can be seen as a sort of generalization of the condition in the well-known Tchebycheff problem: find the monic polynomial of degree n deviating least from zero on $[-1, 1]$ in Tchebycheff-norm.

The deeper reason for the chosen normalization lies in the possibility to derive a result on uniqueness and convergence in an n-dimensional Walsh-array from those in the corresponding n-dimensional Padé-table. Uniqueness is ascertained in the Walsh case by the requirement that we have a so called Tchebycheff-system and it is precisely the chosen normalization that leads to such a system in our case (cf. Lemma 3.2).

In the n-dimensional Padé-table the sequence of cells $(0, 0, \ldots, 0)$, $(1, 1, \ldots, 1)$, $(2, 2, \ldots, 2)$, $\ldots$, $(\lambda, \lambda, \ldots, \lambda)$, $\ldots$ is called the *main diagonal* and the cells satisfying $\mu_1+\mu_2+\ldots+\mu_n=k$ *the plane of approximation of order* k. Now a sequence of cells will be called *regular* if any two arbitrary consecutive cells are adjacent and if the second cell lies in a plane of approximation of order one higher then the order of the first one. The recurrence relations for Padé n-tuples in an n-dimensional Padé-table constitute a generalization of the continued fraction algorithms connected with regular sequences in the ordinary Padé-table. Research on these algorithms closely related to the Jacobi-Perron algorithms (investigated by O. PERRON [8], [9] published thirteen years after H. PADÉ [7]) is carried on by the first mentioned author of this paper.

Now consider the two-dimensional case. Let $S_1, S_2 \in F$ with $|S_1|_F = |S_2|_F = 1$ and assume $(V_{\mu\nu}^{(1)}, V_{\mu\nu}^{(2)})$ to be the Padé-pair for the square (μ, ν). Then $|S_1 V_{\mu\nu}^{(1)} + S_2 V_{\mu\nu}^{(2)}|_F = e^{-\mu-\nu-1}$ or equivalently $S_1 V_{\mu\nu}^{(1)} + S_2 V_{\mu\nu}^{(2)} = 0(x^{\mu+\nu+1})$. Dividing both members by $S_2(x)$ and putting $C(x) = S_1(x)/S_2(x)$ we obtain

$$C(x) V_{\mu\nu}^{(1)}(x) + V_{\mu\nu}^{(2)}(x) = 0(x^{\mu+\nu+1}).$$

So $P_{\mu\nu}(x) = -V_{\mu\nu}^{(2)}(x)/V_{\mu\nu}^{(1)}(x)$ is the Padé-approximant on the square (μ, ν) of the ordinary Padé-table for $C(x)$.

In the metric space F every sequence $\{P_{\mu\nu}(x)\}$ of Padé-approximants to $C(x)$ converges to $C(x)$ if $\mu + \nu \to \infty$. If x is an element of a certain subset of $\mathbb{C}$ however the question of convergence (and uniform convergence) on this set is much more complicated.

For power series $C(x) = \sum_{n=0}^{\infty} c_n x^n$ where $\{c_n\}_{n=0}^{\infty}$ is a so-called *Stieltjes-sequence* fairly detailed results are available; see for instance G. A. BAKER Jr. etc. [1], H. S. WALL [15]. Also the work of A. F. BEARDON [2], [3] and J. S. R. CHISHOLM [4] should be mentioned.

Regarding the pointwise and uniform convergence for sequences $\{P_{\mu\nu}(x)\}$ in which μ and ν augment by unity alternately (so called *step-lines*, examples of regular sequences) the work of the first author of this paper (to appear) is of some interest. For all known functions with a normal table the Padé-approximants on the steplines appear to be approximants of continued fractions which converge to the function on the whole region of convergence of its Taylor series around the origin; this implies convergence of the sequence of Padé-approximants on the diagonals.

2. Consider expressions of the form $W_1 S_1 + W_2 S_2 + \ldots + W_n S_n$ in which $S_1, S_2, \ldots, S_n$ are power series with real coefficients in the real variable x with radius of convergence $\varrho_i > 0$ $(i = 1, 2, \ldots, n)$ respectively and $W_1, W_2, \ldots, W_n$ polynomials with real coefficients of degree $\mu_1, \mu_2, \ldots, \mu_n$ respectively; let $[-a, a]$ be a closed finite interval with $0 < a < \min\{\varrho_1, \varrho_2, \ldots, \varrho_n\}$.

We now want to minimize the expression

$$\max_{|x| \leqslant a} |W_1 S_1 + W_2 S_2 + \ldots + W_n S_n|$$

under condition C for the n-tuple $W_1, W_2, \ldots, W_n$.

In order to be able to solve the problem and maybe even find a unique solution it must be reformulated and some conditions must be imposed upon $S_1, S_2, \ldots, S_n$. These conditions are laid down in the concept of a *Tchebycheff-system* (see S. KARLIN, W. J. STUDDEN [6]), which can also be used in much more general "minimax" problems.

The problem can be reformulated as:

$$\text{minimize} \max_{|x| \leqslant a} \left| \frac{x^{\mu_1}}{\mu_1 !} S_1 - (-W_1^* S_1 - W_2 S_2 - \ldots - W_n S_n) \right| ,$$

where degree $W_1^* = \mu_1 - 1$, degree $W_k = \mu_k$ $(k = 2, 3, \ldots, n)$.
(If $\mu_1 = 0$ we minimize $\max_{|x| \leqslant a} |S_1 - (-W_2 S_2 - \ldots - W_n S_n)|$).

Definition 2.1. A system $\{u_0, u_1, \ldots, u_m\}$ of continuous real-valued functions on a closed finite interval $[a, b]$ is called a *Tchebycheff-system* (T-system) when the determinants

$$\begin{vmatrix} u_0(t_0) & u_0(t_1) & \ldots & u_0(t_m) \\ u_1(t_0) & u_1(t_1) & \ldots & u_1(t_m) \\ \vdots & \vdots & & \vdots \\ u_m(t_0) & u_m(t_1) & \ldots & u_m(t_m) \end{vmatrix}$$

are strictly positive whenever $a \leqslant t_0 < t_1 < \ldots < t_m \leqslant b$.

Example: In the case $u_i(t) = t^i$ $(i = 0, 1, \ldots, m)$ the determinants reduce to familiar Vandermonde determinants from which the positiveness follows for an arbitrary finite interval.

It will be convenient for the sequel to use a slightly different way of ascertaining the T-system property:

Theorem 2.1. If a system $\{u_i\}_{i=0}^m$ of continuous real-valued functions on $[a, b]$ satisfies the condition that every non-trivial (i.e. $\not\equiv 0$) function $\alpha_0 u_0 + \alpha_1 u_1 + \ldots + \alpha_m u_m$ $(\alpha_i \in \mathbf{R};\ i = 0, 1, \ldots, m)$ has at most m zeros on $[a, b]$ then $\{u_i\}_{i=0}^m$ is a T-system except possibly for the sign of one of the functions.

(For the proof see S. Karlin, W. J. Studden [6], theorem 4.1, page 22).

Theorem 2.2. The function $\alpha_0 u_0 + \alpha_1 u_1 + \ldots + \alpha_m u_m$ minimizing

$$\max_{|x| \leqslant a} |f(x) - \sum_{i=0}^{m} \alpha_i u_i(x)|$$

where f is a continuous real-valued function on $[-a, a]$ is uniquely determined if and only if the functions $\{\varepsilon u_0, u_1, \ldots, u_m\}$ constitute a T-system for appropriate $\varepsilon = +1$ or -1.

(Proof: S. Karlin, W. J. Studden [6], theorem 1.1, page 280).

Bearing this in mind we now assume $\{S_1, xS_1, \ldots, x^{\mu_1 - 1} S_1, S_2, \ldots, x^{\mu_2} S_2, \ldots, \ldots, S_n, \ldots, x^{\mu_n} S_n\}$ to be a T-system on $[-a, a]$. In this case the minimizing problem posed at the beginning of this section has a unique solution and placing this unique n-tuple in the cell $(\mu_1, \mu_2, \ldots, \mu_n)$ of an n-dimensional array we arrive at what will be called an *n-dimensional Walsh-array*. For $n = 2$, $S_2 = 1$, we arrive at a slightly different form of the ordinary Walsh-array ([17]). Convergence properties in the ordinary Walsh-array have been considered by E. B. Saff [12], [13], [14] and J. L. Walsh [18].

3. Consider the functions $S_k = e^{w_k x}$ $(k=1, 2, \ldots, n)$; $w_1, w_2, \ldots, w_n \in \mathbb{C}$ pairwise distinct and let G be a bounded subset of $\mathbb{C}$; $\mu_1, \mu_2, \ldots, \mu_n$ non-negative integers.

According to theorem 1.1 there exists a unique n-tuple of polynomials $P_k(\mu_1, \mu_2, \ldots, \mu_n; x)$ $(k=1, 2, \ldots, n)$ satisfying condition C and:

i) $|\sum_{k=1}^{n} P_k(\mu_1, \mu_2, \ldots, \mu_n; x) e^{w_k x}|_F = e^{-\mu_1-\mu_2-\ldots-\mu_n-n+1}$

ii) degree $P_k(\mu_1, \mu_2, \ldots, \mu_n; x) = \mu_k$ $(k=1, 2, \ldots, n)$.

These polynomials and the sum in i) can be given explicitly using H. Jager [5], chapter IV, § 2 with $m=n$, $\varrho_k = \mu_k + 1$ $(k=1, 2, \ldots, n)$. For the sequel we only need:

$$R(x) = \sum_{k=1}^{n} P_k(\mu_1, \mu_2, \ldots, \mu_n; x)\, e^{w_k x} =$$

$$= \frac{\prod_{\nu=2}^{n} (w_1 - w_\nu)^{\mu_\nu+1}}{2\pi i} \int_{|z|=R} \frac{e^{xz}}{\prod_{k=1}^{n} (z - w_k)^{\mu_k+1}} \frac{dz}{z}$$

where R is a fixed real number satisfying:

$$R > 2 \max\,[|w_1|, |w_2|, \ldots, |w_n|, |w_1 - w_2|, |w_1 - w_3|, \ldots, |w_1 - w_n|, 1].$$

We are now able to formulate the following theorem which will be proved by estimating $R(x)$:

Theorem 3.1.

$$\lim_{\mu_1+\mu_2+\ldots+\mu_n\to\infty} \sum_{k=1}^{n} P_k(\mu_1, \mu_2, \ldots, \mu_n; x)\, e^{w_k x} = 0$$

uniformly in x on G.

Proof: Put $\mu_1 + \mu_2 + \ldots + \mu_n = \sigma$ and let M be a bound for G, i.e. for all $x \in G$: $|x| \leqslant M$.

Then:

$$|R(x)| \leqslant \frac{(\max\{|w_1-w_2|, |w_1-w_3|, \ldots, |w_1-w_n|, 1\})^{\sigma+n-1-\mu_1}}{2\pi} \cdot \frac{2\pi R e^{MR}}{(\frac{1}{2}R)^{\sigma+n}\cdot R} \leqslant$$

$$\leqslant d^{\sigma+n} \cdot e^{MR} \text{ where } d = \frac{\max\{|w_1-w_2|, |w_1-w_3|, \ldots, |w_1-w_n|, 1\}}{\frac{1}{2}R}$$

satisfying $0 < d < 1$, which shows the uniform convergence for $\sigma \to \infty$.

Remark: This theorem still holds when we use the aforementioned condition on the first non-zero coefficient of $R(x)$ with $\alpha = 1$ instead of condition C. The difference is that we have to omit the scalar $\prod_{\nu=2}^{n} (w_1 - w_\nu)^{\mu_\nu+1}$ in the formula for $R(x)$ and we can omit $\max\{|w_1-w_2|, |w_1-w_3|, \ldots, |w_1-w_n|\}$ in the lower bound for R and in the proof of the theorem.

We now want to formulate a minimax problem involving the functions $e^{w_k x}$ $(k=1, 2, \ldots, n)$ and therefore we restrict ourselves to the case $w_k \in \mathbf{R}$ $(k=1, 2, \ldots, n)$ and G a real interval $[-a, a]$, $a>0$.

Lemma 3.2. The problem of minimizing

$$\max_{|x|\leqslant a} \left|0-\sum_{k=1}^{n} A_k(\mu_1, \mu_2, \ldots, \mu_n; x)\, e^{w_k x}\right|$$

where the minimum is taken over all polynomial n-tuples with degree $A_k=\mu_k$ $(k=1, 2, \ldots, n)$ satisfying condition C has a unique solution; this solution will be referred to as $\{W_k(\mu_1, \mu_2, \ldots, \mu_n; x)\}_{k=1}^{n}$.

Once this lemma is proven we arrive at the following important theorem:

Theorem 3.2.

$$\lim_{\mu_1+\mu_2+\ldots+\mu_n\to\infty} \sum_{k=1}^{n} W_k(\mu_1, \mu_2, \ldots, \mu_n; x)\, e^{w_k x}=0$$

uniformly in x on $[-a, a]$.

Remark: If μ_1 is fixed the theorem still holds when $\sigma\to\infty$, showing the convergence to be a non-trivial matter.

Proof of theorem 3.2: Since the Padé polynomials belong to the set of n-tuples over which the minimum is taken, lemma 3.1 leads to:

$$\left|\sum_{k=1}^{n} W_k(\mu_1, \mu_2, \ldots, \mu_n; x)\, e^{w_k x}\right| \leqslant \max_{|x|\leqslant a}\left|\sum_{k=1}^{n} W_k(\mu_1, \mu_2, \ldots, \mu_n; x)\, e^{w_k x}\right| \leqslant$$
$$\leqslant \max_{|x|\leqslant a}\left|\sum_{k=1}^{n} P_k(\mu_1, \mu_2, \ldots, \mu_n; x)\, e^{w_k x}\right| < d^{\sigma+n}\cdot e^{MR}$$

$(0<d<1$, M and R as in theorem 3.1); the last expression converging to zero for $\sigma\to\infty$.

Proof of lemma 3.2: Using the same method for reformulating the minimax problem as in section 2 we only have to prove that

$$\{e^{w_1 x}, xe^{w_1 x}, \ldots, x^{\mu_1-1} e^{w_1 x}, e^{w_2 x}, \ldots, x^{\mu_2} e^{w_2 x}, \ldots, e^{w_n x}, \ldots, x^{\mu_n} e^{w_n x}\}$$

is a T-system on $[-a, a]$ except possibly for the sign of one of the functions and then use theorem 2.2. (If $\mu_1=0$, consider the system $\{e^{w_2 x}, \ldots, \ldots, x^{\mu_2} e^{w_2 x}, \ldots, e^{w_n x}, \ldots, x^{\mu_n} e^{w_n x}\}$).

A linear combination with real coefficients of the $\sigma+n-1$ functions of the system gives rise to an exponential sum of degree at most $\{(\mu_1-1)+ +1+\sum_{k=1}^{n}(\mu_k+1)\}-1=\sigma+n-2$. According to G. Polya, G. Szego [11], page 48, problem 75 an exponential sum has at most as many real zeros as its order, so the linear combination of the $\sigma+n-1$ functions has at most $\sigma+n-2$ zeros on $[-a, a]$ and thus the functions form a T-system (except possibly for the sign of one of them) in view of theorem 2.1.

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